Solve system of simultaneous equations with 150 variables (Solution)

150周年校慶

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1. $$\begin{cases} x_1 + x_2 + \cdots + x_k + \cdots + x_{150} = 0 & \ldots\ldots(1) \\ x_1 - x_2 = x_2 - x_3 = \cdots = x_{k-1} - x_k = \cdots = x_{149} - x_{150} = 1 & \ldots\ldots(2) \end{cases}$$

From (2), $(x_1 - x_2) + (x_2 - x_3) + \cdots + (x_{k-1} - x_k) = k - 1$

$$x_1 - x_k = k - 1$$

$$x_k = x_1 - k + 1 \qquad \ldots\ldots(3)$$

(3) ↓ (1), $x_1 + (x_1 - 1) + (x_1 - 2) + \cdots + (x_1 - 149) = 0$

$$150x_1 = 1 + 2 + \cdots + 149 = \frac{149 \times 150}{2}$$

$$x_1 = 74.5,\ x_2 = 73.5,\ \ldots.,\ x_{75} = 0.5,\ x_{76} = -0.5, \ldots., x_{150} = -64.5\ .$$

In general, $x_k = 75.5 - k$

2. $$\begin{cases} x_1 + 2x_2 + 3x_3 + \cdots + 150x_{150} = a_1 & \ldots\ldots(1) \\ x_2 + 2x_3 + 3x_4 + \cdots + 150x_1 = a_2 & \ldots\ldots(2) \\ x_3 + 2x_4 + 3x_5 + \cdots + 150x_2 = a_3 & \ldots\ldots(3) \\ \ldots\ldots\ldots\ldots \\ x_{150} + 2x_1 + 3x_2 + \cdots + 150x_{149} = a_{150} \end{cases}$$

Let $s = x_1 + x_2 + x_3 + \cdots + x_{150}$

Adding all equations of the given system,

$$s + 2s + 3s + \cdots + 150s = a_1 + a_2 + \cdots + a_{150}$$

$$\frac{150(1+150)}{2} s = a_1 + a_2 + \cdots + a_{150}$$

$$s = \frac{1}{11325}(a_1 + a_2 + \cdots + a_{150}) = A \qquad \text{(for simplicity)}$$

(1) – (2), $x_1 + x_2 + x_3 + \cdots + x_{150} - 150x_1 = a_1 - a_2$

$$150x_1 = A + a_2 - a_1$$

$$x_1 = \frac{1}{150}(A + a_2 - a_1)$$

Similarly, (3) – (2), we get $x_2 = \frac{1}{150}(A + a_3 - a_2)$, ……

Continue in this way, lastly, we get $x_{149} = \frac{1}{150}(A + a_{150} - a_{149})$

and $x_{150} = \frac{1}{150}(A + a_1 - a_{150})$

3. $$\begin{cases} x_1 + x_2 + \cdots + x_{150} = 1 \\ x_1 + x_3 + \cdots + x_{150} = 2 \\ x_1 + x_2 + x_4 + \cdots + x_{150} = 3 \\ \dots\dots\dots \\ x_1 + x_2 + \cdots + x_{149} = 150 \end{cases}$$

Let $s = x_1 + x_2 + x_3 + \cdots + x_{150} = 1$

Then the system of equations become

$$s - x_2 = 2,\ s - x_3 = 3, \dots, s - x_{150} = 150$$

Since $s = 1$, $\therefore\ x_2 = -1,\ x_3 = -2, \dots, x_{150} = -149$

Since $x_1 + x_2 + x_3 + \cdots + x_{150} = 1$

$x_1 - 1 - 2 - 3 - \cdots - 149 = 1$

$$x_1 = 1 + \frac{149(149+1)}{2} = 11176$$

4. $$\begin{cases} x_1 - x_2 - x_3 \dots - x_{150} = 2k \\ -x_1 + 3x_2 - x_3 - \cdots - x_{150} = 4k \\ -x_1 - x_2 + 7x_3 - \cdots - x_{150} = 8k \\ \dots\dots\dots \\ -x_1 - x_2 - x_3 - \cdots + (2^{150} - 1)x_{150} = (2^{150})k \end{cases}$$

Let $s = x_1 + x_2 + x_3 + \cdots + x_{150}$

Then the given system of equations become :

$$\begin{cases} -s + 2x_1 = 2k \\ -s + 4x_2 = 4k \\ -s + 8x_3 = 8k \\ \dots\dots\dots \\ -s + 2^{150}x_{150} = 2^{150}k \end{cases}$$

Hence, $x_1 = k + \frac{s}{2},\ x_2 = k + \frac{s}{4},\ x_3 = k + \frac{s}{8}, \dots,\ x_{150} = k + \frac{s}{2^{150}}$

Adding these equalities, $s = 150k + s\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{150}}\right) = 150k + s\left(1 - \frac{1}{2^{150}}\right)$

$$s = 2^{150}(150k)$$

$$x_1 = k + \frac{s}{2} = k(1 + 2^{149}150)$$

$$x_2 = k + \frac{s}{4} = k(1 + 2^{148}150)$$

….

$$x_{150} = k + \frac{s}{2^{150}} = k(1 + 150) = 151k$$

5. $\begin{cases} x_1 + x_2 = 1 \\ x_2 + x_3 = 2 \\ x_3 + x_4 = 3 \\ \dots\dots\dots \\ x_{149} + x_{150} = a_{149} \\ x_{150} + x_1 = a_{150} \end{cases}$

Let $x_1 = k$ (where k is any arbitrary constant)

$$\begin{cases} x_1 = k \\ x_2 = a_1 - x_1 = a_1 - k \\ x_3 = a_2 - x_2 = a_2 - a_1 + k \\ x_4 = a_3 - x_3 = a_3 - a_2 + a_1 - k \\ \dots\dots\dots \\ x_{150} = a_{149} - a_{148} + a_{147} - \dots - a_2 + a_1 - k \end{cases} \quad \dots\dots\text{(i)}$$

Substitute x_1, x_{150} into the last equation of the given system of equation, we have

$(a_{149} - a_{148} + a_{147} - \dots - a_2 + a_1 - k) + k = a_{150}$

$a_1 + a_3 + \cdots + a_{149} = a_2 + a_4 + \cdots + a_{150}$ ……(ii)

So we have two cases:

(a) If (ii) holds the system of equations has unique solution given by (i) .

(b) If (ii) does not hold, the system of equations has no solution.

6. $\begin{cases} x_1x_2 \dots x_{150} = 1 & \dots\dots(1) \\ x_1 - x_2x_3 \dots x_{150} = 1 & \dots\dots(2) \\ x_1x_2 - x_3x_4 \dots x_{150} = 1 & \dots\dots(3) \\ \dots\dots\dots \\ x_1x_2 \dots x_{149} - x_{150} = 1 \end{cases}$

(2)$\times x_1$, $\quad {x_1}^2 - x_1x_2x_3 \dots x_{150} = x_1$

From (1), $\quad {x_1}^2 - 1 = x_1$ or ${x_1}^2 - x_1 - 1 = 0$ or $x_1 = \frac{1\pm\sqrt{5}}{2}$ … (i)

(3) $\times x_1x_2$, $\quad (x_1x_2)^2 - x_1x_2x_3 \dots x_{150} = x_1x_2$

From (1),$(x_1x_2)^2 - 1 = (x_1x_2)$ or $(x_1x_2)^2 - (x_1x_2) - 1 = 0$ or $x_1x_2 = \frac{1\pm\sqrt{5}}{2}$ …(ii)

Similarly, we get $\quad x_1x_2x_3 = \frac{1\pm\sqrt{5}}{2}$ …(iii)

Continue in this way, we get finally, $\quad x_1x_2 \dots x_{149} = \frac{1\pm\sqrt{5}}{2}$

$\frac{\text{(ii)}}{\text{(i)}}$, $\quad x_2 = 1$ or $\frac{\frac{1\pm\sqrt{5}}{2}}{\frac{1\mp\sqrt{5}}{2}} = -\frac{3\pm\sqrt{5}}{2}$

$\frac{\text{(iii)}}{\text{(ii)}}$, $\quad x_3 = 1$ or $\frac{\frac{1\pm\sqrt{5}}{2}}{\frac{1\mp\sqrt{5}}{2}} = -\frac{3\pm\sqrt{5}}{2}$

Continue in this way, we get, $x_2 = x_3 = \cdots = x_{149} = 1$ or $-\frac{3\pm\sqrt{5}}{2}$

To find x_{150}, multiply the last equation of the given system by x_{150}, and using (1), we get $x_1x_2 \dots x_{149}x_{150} - (x_{150})^2 = x_{150}$ or $1 - (x_{150})^2 = x_{150}$

$(x_{150})^2 + x_{150} - 1 = 0$

$\therefore \quad x_{150} = \frac{-1\pm\sqrt{5}}{2}$.

There are many possible combinations of solution, for example, one solution may be :

$(x_1, x_{2,} \dots, x_{149}, x_{150}) = \left(\frac{1+\sqrt{5}}{2}, 1,1, \dots ,1, \frac{-1+\sqrt{5}}{2}\right)$

7.
$$\begin{cases} x_0x_1x_2 = x_0 + x_1 + x_2 & \dots\dots (1) \\ x_1x_2x_3 = x_1 + x_2 + x_3 & \dots\dots (2) \\ \dots\dots\dots & \\ x_{149}x_{150}x_0 = x_{149} + x_{150} + x_0 & \\ x_{150}x_0x_1 = x_{150} + x_0 + x_1 & \end{cases}$$

(1) – (2), $x_1x_2(x_0 - x_3) = x_0 - x_3$

$\therefore \quad x_1x_2 = 1$ or $x_0 = x_3$

If $x_1x_2 = 1$, then from (1), $x_0 = x_0 + x_1 + x_2$ or $x_1 + x_2 = 0$

But $x_1x_2 = 1$ and $x_1 + x_2 = 0$ has no real solutions. (x_1, x_2 are roots of $t^2 + 1 = 0$)

$\therefore \quad x_0 = x_3$.

Similarly, $x_1 = x_4, x_2 = x_5, \dots, x_{148} = x_0,\ x_{149} = x_1, x_{150} = x_2$.

On checking,

$x_1 = x_4 = x_7 = \cdots = x_{148} = x_0 = x_3 = x_6 = \cdots = x_{150} = x_2 = x_5 = \cdots = x_{149}$

$\therefore$ All x_i are equal and let them be equal to x .

From (1), we get $x^3 = 3x$.

$\therefore \quad x = 0$ or $\pm\sqrt{3}$.